SCATTERING THEORY MADE SIMPLE

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Abstract. I will give a brief description of the main issues in scattering theory. Then I will describe a new proof of a classical result in the field. This problem, proving Asymptotic Completeness for two body scattering in QM has many proofs, going back to the 1940's. The new proof is simple, yet substantially more general. It therefore applies to many new problems which were out of reach until now.

1. Introduction

Scattering Theory is the study of the asymptotic behavior of systems as time goes to plus or minus infinity. This is one of the most powerful tools in understanding evolution equations in Science.

It is based on the following two meta-theorems: First, every system has a simple asymptotic dynamics.

Second, it is **sufficient** to find all asymptotic states in order to know everything about the system.

The fundamental example is the process of "seeing".

The observed object plus the light shining on it, is the system. For times large, positive or negative, the light wave (wave satisfying Maxwell equations) are freely moving waves. At times of interactions with the object, they are very complicated to describe. Yet, our eye-brain system can extract from the outgoing waves all the information about the observed object.

Not surprisingly, it was a field of study by Leonardo da Vinci. Here is how he described it:(1450's)

The air that is between bodies is full of the intersections formed by the radiating images of these bodies.

`http://www.antoniosiber.org/da_vinci_i_difuzno_rasprsenje_en.html`

Most of the mathematical analysis centered on classical EM waves. Sommerfeld (1912) introduced,following similar ideas of V. Igantowski(1905) in the context of EM scattering, the notion of outgoing/incoming, but in terms of the behavior for large distance.

A major breakthrough came from Rutherford, who in 1907 wrote one of the great classical papers: it included a new theory, the scattering of a classical particle on another, solving the scattering of Coulomb potential two body problem, introducing new concepts, and doing an experiment to find the structure of an atom. By analysing the outgoing distribution of α particle on thin gold plate, he proved that the atom has a nuclei surrounded by electrons moving far away. He got the Noble prize for that, and is called the father of the nuclear generation.

His work turned scattering theory to the key tool in physics, chemistry, biology and more.

2010 *Mathematics Subject Classification.* 35Q55,

A.Soffer is supported in part by Simons Foundation Grant number 851844 .

Quantum Mechanics posed a new challenge: we now need scattering theory for complex valued waves, following the dynamics of Schrödinger equation.

Formulating the meta-theorem A of scattering, *Möller* (1945) Introduced the idea of scattering wave operators:

$$\Omega = \lim_{t \to \infty} U(-t) U_0(t),$$

where $U(t)$ is the dynamics of the full complicated system, taking the state at time 0 to time t. Similarly, $U_0(t)$ is the dynamics which is simple. Proving that the above limit exists for Ω^* is called Asymptotic Completeness, and is known as the *hard direction.*

So, if you take a state forward in time under the free flow for time t, and then backward under the full flow, you need to have the limit to exist as t goes to infinity. The existence of this limit and its adjoint are the most intricate part of mathematical scattering theory.

Friedrichs (1948) was the fist to prove such results for systems which are obtained as a small perturbations of the simple dynamics (free waves).

More comprehensive results, which applied to concrete models of QM were developed in the 1950's by Cook, and mostly Kato. That was followed by Koruda, Birman and many others.

A new approach, mostly for hyperbolic equations was developed by Lax-Philips (1967). Other works were done by Agmon.

Then in 1977, Enss introduced a new approach, based on time dependent methods, rather than the study of the Green's function of the equation. This was followed by abstract theory of Mourre (1979-1981). A simplified version of Enss method was done by EB Davies(1980). All the above methods could not be used to prove Asymptotic Completeness for N-body Quantum Scattering beyond three particles (Faddeev 1963).

A new approach, which will also be used in this article was developed in 1985 (Sigal-Soffer); it allowed the solution of the N-body scattering problem.

2. The Two Body Scattering Problem

The two body quantum problem can be reduced to a one body scattering problem, where the x coordinate corresponds to the distance between the two particles. We are thus led to consider the following hamiltonian H on $L^2(\mathbb{R}^n)$. Here $H = -\Delta + V(x)$. We assume for simplicity that the potential V is uniformly bounded and decays at infinity like $< x >^{-\mu}, \mu > 1$. The corresponding Evolution equation, the Schrödinger equation is then:

$$i \frac{\partial \phi(t)}{\partial t} = H \phi(t).$$

The initial data and the solution are in the Hilbert space $L^2(\mathbb{R}^n)$.

The scattering problem is then to prove the convergence in an appropriate sense(strong limit) of the following:

$$\Omega^* \phi = s - lim_{t \to \infty} e^{-i\Delta t} e^{-iHt} \phi \tag{2.1}$$

$$\phi = P_c(H) \phi \tag{2.2}$$

$$U_0(-t) \equiv e^{-i\Delta t} \tag{2.3}$$

$$U(t) \equiv e^{-iHt} \tag{2.4}$$

The first key observation is that it is sufficient to prove the limit for a **channel wave operator**, where the full dynamics is localized in the phase space:

$$\Omega^* \phi = s - lim_{t\to\infty} e^{-i\Delta t} J(x,p,t) e^{-iHt}\phi \tag{2.5}$$

$$\lim_{t\to\infty} \|(Id - J(x,p,t))e^{-iHt}\phi\| = 0. \tag{2.6}$$

We now choose

$$J = F_1(\frac{|x-2pt|}{t^\alpha}) = U_0(-t)F_1(\frac{|x|}{t^\alpha})U_0(t), \quad F_1(z) = 1 \text{ for } z<1, \text{zero for } z>2.$$

F_1 is chosen to be smooth and monotonic decreasing.

Next, we will write the wave operator as an integral of derivative of $\Omega_J^*(t)$ and estimate the resulting integral in the absolute value sense:

$$(u, [\Omega_J^*(T) - \Omega_J^*(S)]\phi) = \int_S^T \partial_t (u, U_0(-t)JU(t)\phi)dt = \tag{2.7}$$

$$\int_S^T (u, U_0(-t)[iH_0 J - iJH + \frac{\partial J}{\partial t}]U(t)\phi)dt = \tag{2.8}$$

$$\int_S^T (u, U_0(-t)\{[iH_0, J] - iJV + \frac{\partial J}{\partial t}\}U(t)\phi)dt \equiv M(S,T). \tag{2.9}$$

We use the following identity which is proved by direct computation:

$$[iH_0, J] + \frac{\partial J}{\partial t} = U_0(t)\{\frac{-\alpha|x|}{t^{1+\alpha}}F_1'\}U_0(-t). \tag{2.10}$$

It follows that

$$M(S,T) = \int_S^T (u, U_0(-t)(-iJV + U_0(t)(\frac{-\alpha|x|}{t^{1+\alpha}}F_1')U_0(-t))U(t)\phi)dt \le \tag{2.11}$$

$$(\int_S^T \|\tilde{J}(t)U_0(t)u\|^2 dt)^{1/2}(\int_S^T \|\tilde{J}(t)U(t)\phi\|^2 dt)^{1/2} + \tag{2.12}$$

$$|\int_S^T (u, U_0(-t)JVU(t)\phi)dt|. \tag{2.13}$$

$$\tilde{J}^2(t) \equiv J(t) = (U_0(t)\frac{\alpha}{\sqrt{t}}\tilde{F}_1' U_0(-t))^2 \tag{2.14}$$

$$\tilde{F}_1^2 = F_1'. \tag{2.15}$$

Therefore, if we can show the integrals above converge absolutely to a finite number, then

$$M(S,T) \le c\|u\|\epsilon(S,T,\phi)\|\phi\| \to 0 \text{ as } T,S\to\infty.$$

Since ϕ is fixed this can be made small independently of u, so we can take the sup over all u with norm 1 (in L^2).

The estimate of the interaction term is reduced to basic estimates with respect to the **free flow**, and therefore easily determine the allowed potentials V.

The estimate of the F_1' term, known as propagation estimate, will be proved by a method described next. It is based on finding monotonic functionals with respect to the full flow, up to integrable corrections in time. The required monotonicity is reduced to proving that certain commutators are positive.

We choose the following family of operators to be our **Propagation Observable** (PROB):

$$J = F_1(\frac{|x - 2pt|}{t^\alpha} \leq 1) = U_0(t)F_1(\frac{|x|}{t^\alpha} \leq 1)U_0(-t).$$

Then, we have

$$\partial_t(\phi(t), J(t)\phi(t)) = (\phi(t), U_0(t)F_1'(\frac{|x|}{t^\alpha})U_0(-t)\phi(t))(-\alpha)t^{-1}+ \tag{2.16}$$

$$(\phi(t), i[V(x), J(t)]\phi(t)). \tag{2.17}$$

As remarked above, the potential term is estimated by bounds on the free flow, and it shows that the last term above is integrable in time. Since the first term on the RHS is non-negative, and the integral over time of the LHS is uniformly bounded, we get the following **Propagation Estimate**, which is in fact the desired estimate:

$$\int_1^\infty \|\tilde{F}_1' U_0(-t)U(t)\phi\|^2 \frac{dt}{t} \leq c\|\phi\|^2. \tag{2.18}$$

Finally, we estimate the potential term. We focus on the easy case when the dimension is 3 or larger. By Cauchy-Schwarz Inequality the potential term is bounded by

$$\|u\|_2 \|F_1(\frac{|x|}{t^\alpha})U_0(-t)V(x)\phi(t)\|_2 \leq c\|F_1(\frac{|x|}{t^\alpha})\|_2 \|U_0(-t)V(x)\phi(t)\|_\infty.$$

Therefore, if for example V is an L^2 function, then $V\phi(t)$ is an integrable function with a bound uniform in time t.

We now use the basic estimate on the free flow:

$$\|\psi(t)\|_\infty = \|U_0(t)\psi(0)\|_\infty \leq ct^{-n/2}\|\psi(0)\|_1.$$

Here n is the dimension. Therefore, we conclude that the above expression is bounded by

$$t^{-n/2}\|F_1(\frac{|x|}{t^\alpha})\|_{L^2} \leq ct^{-n(1/2-\alpha/2)}.$$

This last expression is integrable in time for all $n \geq 3$ provided we choose α sufficiently small. Hence we proved the existence of the limit defining the channel wave operator.

The next task is to identify, explicitly as possible, the subspace of initial conditions leading to a free asymptotic wave. In the case of time independent potential scattering it is expected to be the subspace generated by the continuous spectral part of H, the hamiltonian. In particular, it is the subspace with energy support non-negative. This subspace can be identified by going to the representation of the Hilbert space where H acts as multiplication by λ, and restricting to vectors with support where $\lambda \geq$, and orthogonal to any point spectrum in this subspace. Since the problem is linear, and it is sufficient to construct the wave operators on a dense set, one can further restrict the set of initial data to vectors with support in $0 < \epsilon \leq \lambda \leq M < \infty$. There are few different ways to show that this subspace leads to asymptotic free waves. Here I will show it by an argument which applies for general scattering problems. To this end we introduce a new PROB. Let $< x >$ be a smoothed version of $|x|$ near the origin. Define the operator γ by

$$i[-\Delta, < x >] = -ig \cdot \nabla - i\nabla \cdot g = g \cdot P + P \cdot g,$$

where $g(x)$ is a bounded smooth vector-field, which is equal to $x/|x|$ for $|x| > 2$. A direct computation then gives:

$$i[-\Delta, F_1(< x > /t^\alpha \geq 1)] = P \cdot \nabla_x F_1 + \nabla_x F_1 \cdot P = t^{-\alpha}(F_1'\gamma + \gamma F_1').$$

Now, consider the following PROB:

$$B = \gamma F_1 + F_1\gamma.$$

We now compute the commutator with H:

$$i[H, B] = i[-\Delta, B] - 2F_1 g(x) \cdot \nabla V(x) = 4t^{-\alpha}\gamma F_1'\gamma + t^{-3\alpha}\tilde{F}_1 + 4F_1 L^2/|x|^3 + \mathcal{O}(t^{-\alpha\mu}). \tag{2.19}$$

$$L^2 = -\Delta_{S^{n-1}} \tag{2.20}$$

$$|V(x)| \leq c < x >^{-\mu} . \tag{2.21}$$

Since both F_1, F_1' are non-negative, the above B is a propagation observable, provided $\alpha > 1/3$, $\alpha\mu > 1$. If we now compute the derivative wrt time of the expectation of B in the state of the system at time t, we get a positive term from the above plus an integrable term, plus a term coming from the derivative of B wrt time. This last term is of the form

$$(\phi(t), F_1'\gamma\phi(t))t^{-1}.$$

As such it is not integrable; however, by C_S inequality it is reduced to integrable quantity times a small number times an integral similar to the leading positive term. So, by redoing the estimate twice, this term is also controlled. See [**?**] for details. It follows that:

$$\lim_{t\to\infty} (\phi(t), (F_1\gamma + \gamma F_1)\phi(t)) = \Gamma < \infty \tag{2.22}$$

$$\int_1^\infty (\phi(t), \gamma F_1'\gamma\phi(t))t^{-\alpha} dt < c\|\phi(t)\|^2_{H^{1/2}}. \tag{2.23}$$

The operator γ is the radial derivative away from the origin. That is the radial velocity. If the limit above is negative, this will lead to blowup in a finite time, which we excluded by the assumption that H is self adjoint, and the initial data is localized in energy. If the limit is zero, this solution is weakly localized. By following similar arguments as above, and using the fact that $i[-\Delta, < x >] = \gamma$, one can prove that for such a state

$$(\phi_{wls}(t), |x|\phi_{wls}(t)) \leq ct^{1/2}.$$

Such a behavior is impossible for initial data in the continuous spectrum, as we will see below.

So, it remains to understand the case of $\Gamma > 0$. The claim is that such a state must have a part which is scattering to a free wave. This is intuitively clear, since this state has a constant radial velocity at infinity. To prove it, note that

$$\lim e^{-i\Delta t} F_1\gamma\phi(t)$$

exists by the above PRES. Furthermore, the limit can not be zero, since the solution has a non-zero limit on the support of $F_1\gamma$. It remains to show that all states in the continuous spectral subspace of H are asymptotically free. For this, it is sufficient to show that on a subsequence of times, the solution has a part in the region $|x| > \delta t$, $\delta > 0$. This follows for example from the following:

$$(1/t)(\phi(t), A\phi(t)) = (1/t)\int_1^t (\phi(t), i[H, A]\phi(t))dt + M/t = \tag{2.24}$$

$$(1/t)\int_1^t (\phi(t), (2H - 2V - x \cdot \nabla V)\phi(t))dt + M/t \geq \tag{2.25}$$

$$2E - \eta(t) \tag{2.26}$$

$$\eta(t) \to 0,\ t \to \infty \tag{2.27}$$

$$E \equiv (\phi(t), H\phi(t)). \tag{2.28}$$

The reason that the potential terms converge to zero as $t \to \infty$ is due to the fact that the above expectations, when written in the representation where H is a multiplication operator, the corresponding Hilbert space consists of functions integrable wrt to a finite continuous measure. Therefore, since the dynamics is given by multiplication by $e^{i\lambda t}$ in this space, the limit is zero for the absolutely continuous part, by Riemann-Lebegue Lemma, and by a Wiener's Lemma that states that the average over t of the Fourier transform of a continuous finite measure vanishes as $t \to \infty$. This last fact also shows that there is NO singular continuous part to the spectrum of the above H.

Let me add some final comments: This new way of studying the scattering problem, generalizes to time dependent potentials and non linear dispersive equations. A follow-up of this approach requires the control of the weakly localized parts for such general dispersive equations. So far, a rather complete results are obtained in the case of NLS with radial data in 3 dimensions; further assumptions are needed, of a technical nature, on the interaction terms. And one must assume that the solution for the given initial data is uniformly bounded in the Sobolev space H^1.

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